# An explicit formula for the intersection of two polynomials of regular languages

Jean-Eric Pin

# An explicit formula for the intersection of two polynomials of regular languages

Jean-Éric Pin[1⋆]

LIAFA, University Paris-Diderot and CNRS, France.

**Abstract.** Let $\mathcal{L}$ be a set of regular languages of $A^*$. An $\mathcal{L}$-polynomial is a finite union of products of the form $L_0a_1L_1 \cdots a_nL_n$, where each $a_i$ is a letter of $A$ and each $L_i$ is a language of $\mathcal{L}$. We give an explicit formula for computing the intersection of two $\mathcal{L}$-polynomials. Contrary to Arfi's formula (1991) for the same purpose, our formula does not use complementation and only requires union, intersection and quotients. Our result also implies that if $\mathcal{L}$ is closed under union, intersection and quotient, then its polynomial closure, its unambiguous polynomial closure and its left [right] deterministic polynomial closure are closed under the same operations.

## 1 Introduction

Let $\mathcal{L}$ be a set of regular languages of $A^*$. An $\mathcal{L}$-*polynomial* is a finite union of products of the form $L_0a_1L_1 \cdots a_nL_n$, where each $a_i$ is a letter of $A$ and each $L_i$ is a language of $\mathcal{L}$. The *polynomial closure* of $\mathcal{L}$, denoted by $\mathrm{Pol}(\mathcal{L})$, is the set of all $\mathcal{L}$-polynomials.

It was proved by Arfi [1] that if $\mathcal{L}$ is closed under Boolean operations and quotient, then $\mathrm{Pol}(\mathcal{L})$ is closed under intersection. This result was obtained by giving an explicit formula for computing the intersection of two polynomials of regular languages.

It follows from the main theorem of [6] that Arfi's result can be extended to the case where $\mathcal{L}$ is only closed under union, intersection and quotient. However, this stronger statement is obtained as a consequence of a sophisticated result involving profinite equations and it is natural to ask for a more elementary proof.

The objective of this paper is to give a new explicit formula for computing the intersection of two $\mathcal{L}$-polynomials. Contrary to the formula given in [1], our formula only requires using union, intersection and quotients of languages of $\mathcal{L}$. Our proof is mainly combinatorial, but relies heavily on the notion of syntactic ordered monoid, a notion first introduced by Schützenberger [14] (see also [10]). The main difficulty lies in finding appropriate notation to state the formula, but then its proof is merely a verification.

Our result also leads to the following result, that appears to be new: if $\mathcal{L}$ is closed under union, intersection and quotient, then its unambiguous polynomial

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closure and its left [right] deterministic polynomial closure are closed under the same operations.

Let us mention also that our algorithm can be readily extended to the setting of infinite words by using syntactic ordered $\omega$-semigroups [8].

## 2 Background and notation

### 2.1 Syntactic order

The *syntactic congruence* of a language $L$ of $A^*$ is the congruence on $A^*$ defined by $u \sim_L v$ if and only if, for every $x, y \in A^*$,

$$xuy \in L \iff xvy \in L$$

The monoid $M = A^*/{\sim_L}$ is the *syntactic monoid* of $L$ and the natural morphism $\eta : A^* \to M$ is called the *syntactic morphism* of $L$. It is a well-known fact that a language is regular if and only if its syntactic monoid is finite.

The *syntactic preorder*[1] of a language $L$ is the relation $\leqslant_L$ over $A^*$ defined by $u \leqslant_L v$ if and only if, for every $x, y \in A^*$, $xuy \in L$ implies $xvy \in L$. The associated equivalence relation is the syntactic congruence $\sim_L$. Further, $\leqslant_L$ induces a partial order on the syntactic monoid $M$ of $L$. This partial order $\leqslant$ is compatible with the product and can also be defined directly on $M$ as follows: given $s, t \in M$, one has $s \leqslant t$ if and only if, for all $x, y \in M$, $xsy \in \eta(L)$ implies $xty \in \eta(L)$. The ordered monoid $(M, \leqslant)$ is called the *syntactic ordered monoid* of $L$.

Let us remind an elementary but useful fact: if $v \in L$ and $\eta(u) \leqslant \eta(v)$, then $u \in L$. This follows immediately form the definition of the syntactic order by taking $x = y = 1$.

### 2.2 Quotients

Recall that if $L$ is a language of $A^*$ and $x$ is a word, the *left quotient* of $L$ by $x$ is the language $x^{-1}L = \{z \in A^* \mid xz \in L\}$. The *right quotient* $Ly^{-1}$ is defined in a symmetrical way. Right and left quotients commute, and thus $x^{-1}Ly^{-1}$ denotes either $x^{-1}(Ly^{-1})$ or $(x^{-1}L)y^{-1}$. For each word $v$, let us set

$$[L]_{\uparrow v} = \{u \in A^* \mid \eta(v) \leqslant \eta(u)\}$$
$$[L]_{=v} = \{u \in A^* \mid \eta(u) = \eta(v)\}$$

[1] In earlier papers [6,10,13], I used the opposite preorder, but it seems preferable to go back to Schützenberger's original definition.

**Proposition 2.1.** *The following formulas hold:*

$$[L]_{\uparrow v} = \bigcap_{\{(x,y)\in A^*\times A^* \mid v\in x^{-1}Ly^{-1}\}} x^{-1}Ly^{-1} \tag{1}$$

$$[L]_{=v} = [L]_{\uparrow v} - \bigcup_{\eta(v)<\eta(u)} [L]_{\uparrow u} \tag{2}$$

$$[L]_{\uparrow v} = \bigcup_{\eta(v)\leqslant\eta(u)} [L]_{=u} \tag{3}$$

*Proof.* A word $u$ belongs to the right hand side of (1) if and only if the condition $v \in x^{-1}Ly^{-1}$ implies $u \in x^{-1}Ly^{-1}$, which is equivalent to stating that $v \leqslant_L u$, or $\eta(v) \leqslant \eta(u)$, or yet $u \in [L]_{\uparrow v}$. This proves (1). Formulas (2) and (3) are obvious. □

Let us make precise a few critical points. First, $v$ always belongs to $[L]_{\uparrow v}$. This is the case even if $v$ cannot be completed into a word of $L$, that is, if $v$ does not belong to any quotient $x^{-1}Ly^{-1}$. In this case, the intersection on the right hand side of (1) is indexed by the empty set and is therefore equal to $A^*$.

Secondly, the intersection occurring on the right hand side of (1) and the union occurring on the right hand side of (2) are potentially infinite, but they are finite if $L$ is a regular language, since a regular language has only finitely many quotients.

## 3 Infiltration product and infiltration maps

The definition below is a special case of a more general definition given in [7]. A word $c_1 \cdots c_r$ belongs to the *infiltration product* of two words $a_1 \cdots a_p$ and $v = b_1 \cdots b_q$, if there are two order preserving maps $\alpha : \{1,\ldots,p\} \to \{1,\ldots,r\}$ and $\beta : \{1,\ldots,q\} \to \{1,\ldots,r\}$ such that

(1) for each $i \in \{1,\ldots,p\}$, $a_i = c_{\alpha(i)}$,

(2) for each $i \in \{1,\ldots,q\}$, $b_i = c_{\beta(i)}$,

(3) the union of the ranges of $\alpha$ and $\beta$ is $\{1,\ldots,r\}$.

For instance, the set $\{ab, aab, abb, aabb, abab\}$ is the infiltration product of $ab$ and $ab$ and the set $\{aba, bab, abab, abba, baab, baba\}$ is the infiltration product of $ab$ and $ba$.

A pair of maps $(\alpha,\beta)$ satisfying Conditions (1)–(3) is called a *pair of infiltration maps*. Note that these conditions imply that $p + q \leqslant r$.

In the example pictured in Figure 1, one has $p = 4$, $q = 2$ and $r = 5$. The infiltration maps $\alpha$ and $\beta$ are given by $\alpha(1) = 1$, $\alpha(2) = 2$, $\alpha(3) = 3$, $\alpha(4) = 4$ and $\beta(1) = 3$, $\beta(2) = 5$.

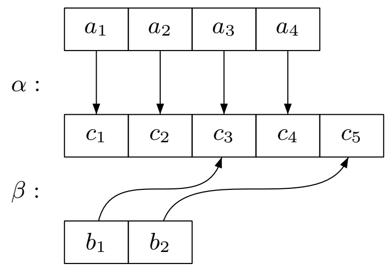



**Fig. 1.** A pair of infiltration maps.

In order to state our main theorem in a precise way, we need to handle the intervals of the form $\{\alpha(i)+1, \ldots, \alpha(i+1)-1\}$, but also the two extremal intervals $\{1, \ldots, \alpha(1)-1\}$ and $\{\alpha(p)+1, \ldots, r\}$. As a means to get a uniform notation, it is convenient to extend $\alpha$ and $\beta$ to mappings $\alpha : \{0, \ldots, p+1\} \to \{0, \ldots, r+1\}$ and $\beta : \{0, \ldots, q+1\} \to \{0, \ldots, r+1\}$ by setting $\alpha(0) = \beta(0) = 0$ and $\alpha(p+1) = \beta(q+1) = r+1$. The two extremal intervals are now of the standard form $\{\alpha(i)+1, \ldots, \alpha(i+1)-1\}$, with $i = 0$ and $i = p$, respectively. Further, we introduce the two maps $\bar{\alpha} : \{0, \ldots, r\} \to \{0, \ldots, p\}$ and $\bar{\beta} : \{0, \ldots, r\} \to \{0, \ldots, q\}$ defined by

$$\bar{\alpha}(i) = \max\{k \mid \alpha(k) \leqslant i\} \quad \text{and} \quad \bar{\beta}(i) = \max\{k \mid \beta(k) \leqslant i\}.$$

For instance, one gets for our example:

$$\begin{array}{llllll}
\bar{\alpha}(0) = 0 & \bar{\alpha}(1) = 1 & \bar{\alpha}(2) = 2 & \bar{\alpha}(3) = 3 & \bar{\alpha}(4) = 4 & \bar{\alpha}(5) = 4 \\
\bar{\beta}(0) = 0 & \bar{\beta}(1) = 0 & \bar{\beta}(2) = 0 & \bar{\beta}(3) = 1 & \bar{\beta}(4) = 1 & \bar{\beta}(5) = 2
\end{array}$$

These two functions are conveniently represented in Figure 2

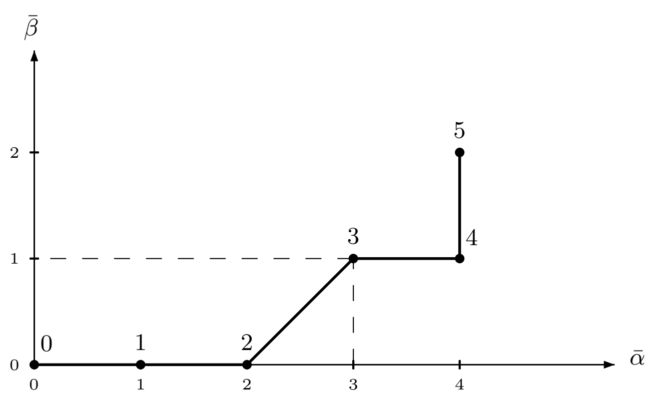



**Fig. 2.** Graphs of $\bar{\alpha}$ and $\bar{\beta}$ : for instance, $\bar{\alpha}(3) = 3$ and $\bar{\beta}(3) = 1$.

The next lemmas summarize the connections between $\alpha$ and $\bar{\alpha}$. Of course, similar properties hold for $\beta$ and $\bar{\beta}$.

**Lemma 3.1.** *The following properties hold:*

1. $\bar{\alpha}(\alpha(k)) = k$, *for* $0 \leqslant k \leqslant p$.
2. $\bar{\alpha}(s+1) \leqslant \bar{\alpha}(s) + 1$, *for* $0 \leqslant s \leqslant r-1$.
3. $k \leqslant \bar{\alpha}(s)$ *if and only if* $\alpha(k) \leqslant s$, *for* $0 \leqslant k \leqslant p$ *and* $0 \leqslant s \leqslant r$.
4. $k \geqslant \bar{\alpha}(s)$ *if and only if* $\alpha(k+1) \geqslant s+1$, *for* $0 \leqslant k \leqslant p-1$ *and* $0 \leqslant s \leqslant r-1$.

*Proof.* These properties follow immediately from the definition of $\bar{\alpha}$. $\square$

**Lemma 3.2.** *For* $0 \leqslant s \leqslant r-1$, *the conditions* $\bar{\alpha}(s+1) = \bar{\alpha}(s) + 1$ *and* $\alpha(\bar{\alpha}(s+1)) = s+1$ *are equivalent.*

*Proof.* Put $k = \bar{\alpha}(s)$ and suppose that $\bar{\alpha}(s+1) = k+1$. Since $k+1 \leqslant \bar{\alpha}(s+1)$, Lemma 3.1 (3) shows that $\alpha(k+1) \leqslant s+1$. Further, since $k \geqslant \bar{\alpha}(s)$, Lemma 3.1 (4) shows that $\alpha(k+1) \geqslant s+1$. Therefore $\alpha(k+1) = s+1$ and finally $\alpha(\bar{\alpha}(s+1)) = s+1$.

Conversely, suppose that $\alpha(\bar{\alpha}(s+1)) = s+1$. Putting $\bar{\alpha}(s+1) = k+1$, one gets $\alpha(k+1) = s+1$ and Lemma 3.1 (4) shows that $k \geqslant \bar{\alpha}(s)$. By Lemma 3.1 (2), one gets $\bar{\alpha}(s+1) \leqslant \bar{\alpha}(s) + 1$ and hence $k \leqslant \bar{\alpha}(s)$. Thus $\bar{\alpha}(s) = k$ and $\bar{\alpha}(s+1) = \bar{\alpha}(s) + 1$. $\square$

Let us denote by $P_\alpha(s)$ the property $\bar{\alpha}(s+1) = \bar{\alpha}(s) + 1$.

**Lemma 3.3.** *For* $0 \leqslant s \leqslant r-1$, *one of* $P_\alpha(s)$ *or* $P_\beta(s)$ *holds.*

*Proof.* Since the union of the ranges of $\alpha$ and $\beta$ is $\{1, \ldots, r\}$, there is an integer $k \geqslant 0$ such that either $\alpha(k+1) = s+1$ or $\beta(k+1) = s+1$. In the first case, one gets $\bar{\alpha}(s+1) = \bar{\alpha}(\alpha(k+1)) = k+1$ and Lemma 3.1 (3) shows that $\bar{\alpha}(s) \leqslant k$. Since $\bar{\alpha}(s+1) \leqslant \bar{\alpha}(s) + 1$ by Lemma 3.1 (2), one also has $k \leqslant \bar{\alpha}(s)$ and finally $\bar{\alpha}(s) = k$, which proves $P_\alpha(s)$. In the latter case, one gets $P_\beta(s)$ by a similar argument. $\square$

## 4 Main result

Let $a_1, \ldots, a_p, b_1, \ldots, b_q$ be letters of $A$ and let $K_0, \ldots, K_p, L_0, \ldots, L_q$ be languages of $A^*$. Let $K = K_0 a_1 K_1 \cdots a_p K_p$ and $L = L_0 b_1 L_1 \cdots b_q L_q$.

A word of $K \cap L$ can be factorized as $u_0 a_1 u_1 \cdots a_p u_p$, with $u_0 \in K_0, \ldots, u_p \in K_p$ and as $v_0 b_1 v_1 \cdots b_q v_q$, with $v_0 \in L_0, \ldots, v_q \in L_q$. These two factorizations can be refined into a single factorization of the form $z_0 c_1 z_1 \cdots c_r z_r$, where $c_1 \cdots c_r$ belongs to the infiltration product of $a_1 \cdots a_p$ and $b_1 \cdots b_q$.

For instance, for $p = 4$ and $q = 2$, one could have $r = 5$, with the relations $c_1 = a_1$, $c_2 = a_2$, $c_3 = a_3 = b_1$, $c_4 = a_4$ and $c_5 = b_2$, leading to the factorization $z_0 c_1 z_1 c_2 z_2 c_3 z_3 c_4 z_4 c_5 z_5$, as pictured in Figure 3.

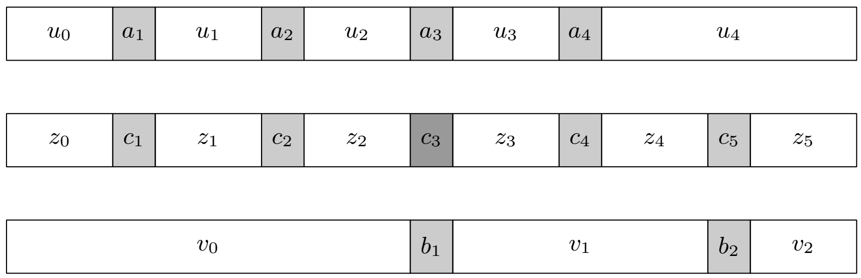



**Fig. 3.** A word of $K \cap L$ and its factorizations.

The associated pair of infiltration maps $(\alpha, \beta)$ is given by

$$\begin{array}{llll} \alpha(1) = 1 & \alpha(2) = 2 & \alpha(3) = 3 & \alpha(4) = 4 \\ \beta(1) = 3 & \beta(2) = 5 & & \end{array}$$

Two series of constraints will be imposed on the words $z_i$:

$$z_0 \in K_0,\ z_1 \in K_1,\ z_2 \in K_2,\ z_3 \in K_3 \text{ and } z_4c_5z_5 \in K_4,$$
$$z_0c_1z_1c_2z_2 \in L_0,\ z_3c_4z_4 \in L_1 \text{ and } z_5 \in L_2.$$

We are now ready to state our main result. Let us denote by $I(p, q)$ the set of pairs of infiltration maps $(\alpha, \beta)$ with domain $\{1, \ldots, p\}$ and $\{1, \ldots, q\}$, respectively. Since $r \leqslant p + q$, the set $I(p, q)$ is finite.

**Theorem 4.1.** *Let $K = K_0a_1K_1 \cdots a_pK_p$ and $L = L_0b_1L_1 \cdots b_qL_q$ be two products of languages. Then their intersection is given by the formulas*

$$K \cap L = \bigcup_{(\alpha,\beta)\in I(p,q)} U(\alpha, \beta) \tag{4}$$

*where*

$$U(\alpha, \beta) = \bigcup_{(z_0,\ldots,z_r)\in C(\alpha,\beta)} U_0c_1U_1 \ \cdots \ c_rU_r \tag{5}$$

*and, for $0 \leqslant i \leqslant r$,*

$$U_i = [K_{\bar\alpha(i)}]_{\uparrow z_i} \cap [L_{\bar\beta(i)}]_{\uparrow z_i} \tag{6}$$

*and $C(\alpha, \beta)$ is the set of $(r + 1)$-tuples $(z_0, \ldots, z_r)$ of words such that*

(C$_1$) *for $0 \leqslant k \leqslant p$, $z_{\alpha(k)}c_{\alpha(k)+1}z_{\alpha(k)+1} \cdots c_{\alpha(k+1)-1}z_{\alpha(k+1)-1} \in K_k$,*

(C$_2$) *for $0 \leqslant k \leqslant q$, $z_{\beta(k)}c_{\beta(k)+1}z_{\beta(k)+1} \cdots c_{\beta(k+1)-1}z_{\beta(k+1)-1} \in L_k$.*

For instance, if $(\alpha, \beta)$ is the pair of infiltration maps of our example, one would have

$$U(\alpha, \beta) = \bigcup_{(z_0,\ldots,z_5)\in C(\alpha,\beta)} ([K_0]_{\uparrow z_0} \cap [L_0]_{\uparrow z_0})a_1([K_1]_{\uparrow z_1} \cap [L_0]_{\uparrow z_1})a_2$$
$$([K_2]_{\uparrow z_2} \cap [L_0]_{\uparrow z_2})b_1([K_3]_{\uparrow z_3} \cap [L_1]_{\uparrow z_3})a_4([K_4]_{\uparrow z_4} \cap [L_1]_{\uparrow z_4})b_2([K_4]_{\uparrow z_5} \cap [L_2]_{\uparrow z_5})$$

and the conditions ($C_1$) and ($C_2$) would be

($C_1$) $z_0 \in K_0$, $z_1 \in K_1$, $z_2 \in K_2$, $z_3 \in K_3$, $z_4c_5z_5 \in K_4$,

($C_2$) $z_0c_1z_1c_2z_2 \in L_0$, $z_3c_4z_4 \in L_1$ and $z_5 \in L_2$.

Before proving the theorem, it is important to note that if the languages $K_0$, $\ldots$, $K_p$, $L_0$, $\ldots$, $L_q$ are regular, the union indexed by $C(\alpha, \beta)$ is actually a finite union. Indeed, Proposition 2.1 shows that, if $R$ is a regular language, there are only finitely many languages of the form $[R]_z$.

*Proof.* Let $U$ be the right hand side of (4). We first prove that $K \cap L$ is a subset of $U$. Let $z$ be a word of $K \cap L$. Then $z$ can be factorized as $u_0a_1u_1 \cdots a_pu_p$, with $u_0 \in K_0$, $\ldots$, $u_p \in K_p$ and as $v_0b_1v_1 \cdots b_qv_q$, with $v_0 \in L_0$, $\ldots$, $v_q \in L_q$. The common refinement of these two factorizations leads to a factorization of the form $z_0c_1z_1 \cdots c_rz_r$, where each letter $c_k$ is either equal to some $a_i$ or to some $b_j$ or both. This naturally defines a pair of infiltration maps $\alpha : \{1, \ldots, p\} \to \{1, \ldots, r\}$ and $\beta : \{1, \ldots, q\} \to \{1, \ldots, r\}$. Conditions ($C_1$) and ($C_2$) just say that the factorization $z_0c_1z_1 \cdots c_rz_r$ is a refinement of the two other ones. Now, since, for $0 \leqslant i \leqslant r$, the word $z_i$ belongs to $[K_{\bar\alpha(i)}]_{\uparrow z_i} \cap [L_{\bar\beta(i)}]_{\uparrow z_i}$, the word $z$ belongs to $U$. Thus $K \cap L \subseteq U$.

We now prove the opposite inclusion. Let $r \leqslant p + q$ be an integer, let $\alpha : \{1, \ldots, p\} \to \{1, \ldots, r\}$ and $\beta : \{1, \ldots, q\} \to \{1, \ldots, r\}$ be two infiltration maps and let $(z_0, \ldots, z_r) \in C(\alpha, \beta)$ and $c_1, \ldots, c_r$ satisfying ($C_1$) and ($C_2$). It suffices to prove that $U_0c_1U_1 \cdots c_rU_r$ is a subset of $K \cap L$. We need a stronger version of ($C_1$) and ($C_2$).

**Lemma 4.2.** *The following relations hold:*

($C_3$) *for* $0 \leqslant k \leqslant p$, $U_{\alpha(k)}c_{\alpha(k)+1}U_{\alpha(k)+1} \cdots c_{\alpha(k+1)-1}U_{\alpha(k+1)-1} \subseteq K_k$,

($C_4$) *for* $0 \leqslant k \leqslant q$, $U_{\beta(k)}c_{\beta(k)+1}U_{\beta(k)+1} \cdots c_{\beta(k+1)-1}U_{\beta(k+1)-1} \subseteq L_k$.

Coming back once again to our main example, these conditions would be

($C_3$) $U_0 \subseteq K_0$, $U_1 \subseteq K_1$, $U_2 \subseteq K_2$, $U_3 \subseteq K_3$, $U_4c_4U_5 \subseteq K_4$,

($C_4$) $U_0c_1U_2c_2U_2 \subseteq L_0$, $U_3c_4U_4 \subseteq L_1$, $U_5 \subseteq L_5$.

*Proof.* Let $\eta_k$ be the syntactic morphism of $K_k$. To simplify notation, let us set $i = \alpha(k) + 1$ and $j = \alpha(k + 1) - 1$. Since $\alpha(k) = i - 1 < i < \cdots < j < \alpha(k + 1)$, one gets $\bar\alpha(i - 1) = \bar\alpha(i) = \ldots = \bar\alpha(j) = k$. Let $u_{i-1} \in U_{i-1}$, $u_i \in U_i$, $\ldots$, $u_j \in U_j$. Then $u_{i-1} \in [U_k]_{\uparrow z_{i-1}}$, $u_i \in [U_k]_{\uparrow z_i}$, $\ldots$, $u_j \in [U_k]_{\uparrow z_j}$ and by definition, $\eta_k(z_{i-1}) \leqslant \eta_k(u_{i-1})$, $\eta_k(z_i) \leqslant \eta_k(u_i)$, $\ldots$, $\eta_k(z_j) \leqslant \eta_k(u_j)$. Therefore we get

$$\begin{aligned}\eta_k(z_{i-1}c_iz_i \cdots c_jz_j) &= \eta_k(z_{i-1})\eta_k(c_i)\eta_k(z_i) \cdots \eta_k(c_j)\eta_k(z_j)\\ &\leqslant \eta_k(u_{i-1})\eta_k(c_i)\eta_k(u_i) \cdots \eta_k(c_j)\eta_k(u_j) = \eta_k(u_{i-1}c_iu_i \cdots c_ju_j)\end{aligned}$$

Now, since $z_{i-1}c_iz_i \cdots c_jz_j \in K_k$ by ($C_1$), we also get $u_{i-1}c_iu_i \cdots c_ju_j \in K_k$, which proves ($C_3$). The proof of ($C_4$) is similar. □

Now, since $\bar\alpha$ and $\bar\beta$ are surjective, Lemma 4.2 shows that $U_0c_1U_1 \cdots c_rU_r$ is a subset of $K \cap L$, which concludes the proof of the theorem. □

*Example 4.3.* Let $K = b^*aA^*ba^*$ and $L = a^*bA^*ab^*$. The algorithm described in Theorem 4.1 gives for $K \cap L$ the expression $aa^*bA^*ba^*a \cup bb^*aA^*ba^*a \cup aa^*bA^*ab^*b \cup bb^*aA^*ab^*b \cup aa^*ba^*a \cup bb^*ab^*b$.

**Corollary 4.4.** *Let $\mathcal{L}$ be a lattice of regular languages closed under quotient. Then its polynomial closure is also a lattice closed under quotient.*

## 5 Some variants of the product

We consider in this section two variants of the product introduced by Schützenberger in [15]: unambiguous and deterministic products. These products were also studied in [2,3,4,5,9,11,12,13].

### 5.1 Unambiguous product

The marked product $L = L_0a_1L_1 \cdots a_nL_n$ of $n$ nonempty languages $L_0$, $L_1$, ..., $L_n$ of $A^*$ is *unambiguous* if every word $u$ of $L$ admits a unique factorization of the form $u_0a_1u_1 \cdots a_nu_n$ with $u_0 \in L_0$, $u_1 \in L_1$, ..., $u_n \in L_n$. We require the languages $L_i$ to be nonempty to make sure that subfactorizations remain unambiguous:

**Proposition 5.1.** *Let $L_0a_1L_1 \cdots a_nL_n$ be an unambiguous product and let $i_1$, ..., $i_k$ be a sequence of integers satisfying $0 < i_1 < \ldots < i_k < n$. Finally, let $R_0 = L_0a_1L_1 \cdots a_{i_1-1}L_{i_1-1}$, $R_1 = L_{i_1}a_{i_1+1}L_1 \cdots a_{i_2-1}L_{i_2-1}$, ..., $R_k = L_{i_k}a_{i_k+1}L_{i_k+1} \cdots a_nL_n$. Then the product $R_0a_{i_1}R_1 \cdots a_{i_k}R_k$ is unambiguous.*

*Proof.* Trivial.

The *unambiguous polynomial closure* of a class of languages $\mathcal{L}$ of $A^*$ is the set of languages that are finite unions of unambiguous products of the form $L_0a_1L_1 \cdots a_nL_n$, where the $a_i$'s are letters and the $L_i$'s are elements of $\mathcal{L}$. The term *closure* actually requires a short justification.

**Proposition 5.2.** *Any unambiguous product of unambiguous products is unambiguous.*

*Proof.* Let

$$\begin{aligned} L_0 &= L_{0,0}a_{1,0}L_{1,0} \cdots a_{k_0,0}L_{k_0,0} \\ L_1 &= L_{0,1}a_{1,1}L_{1,1} \cdots a_{k_1,1}L_{k_1,1} \\ &\vdots \\ L_n &= L_{0,n}a_{1,n}L_{1,n} \cdots a_{k_n,n}L_{k_n,n} \end{aligned} \tag{7}$$

be unambiguous products and let $L = L_0b_1L_1 \cdots b_nL_n$ be an unambiguous product. We claim that the product

$$L_{0,0}a_{1,0}L_{1,0} \cdots a_{k_0,0}L_{k_0,0}b_1L_{0,1}a_{1,1}L_{1,1} \cdots b_nL_{0,n}a_{1,n}L_{1,n} \cdots a_{k_n,n}L_{k_n,n}$$

is unambiguous. Let $u$ be a word of $L$ with two factorizations

$$x_{0,0}a_{1,0}x_{1,0}\cdots a_{k_0,0}x_{k_0,0}b_1x_{0,1}a_{1,1}x_{1,1}\cdots b_nx_{0,n}a_{1,n}x_{1,n}\cdots a_{k_n,n}x_{k_n,n}$$

and

$$y_{0,0}a_{1,0}y_{1,0}\cdots a_{k_0,0}y_{k_0,0}b_1y_{0,1}a_{1,1}x_{1,1}\cdots b_ny_{0,n}a_{1,n}y_{1,n}\cdots a_{k_n,n}y_{k_n,n}$$

with $x_{0,0}, y_{0,0} \in L_{0,0}, \ldots, x_{k_n,n}, y_{k_n,n} \in L_{k_n,n}$. Setting

$$\begin{aligned} x_0 &= x_{0,0}a_{1,0}x_{1,0}\cdots a_{k_0,0}x_{k_0,0} & y_0 &= y_{0,0}a_{1,0}y_{1,0}\cdots a_{k,0}y_{k_0,0} \\ x_1 &= x_{0,1}a_{1,1}x_{1,1}\cdots a_{k_1,1}x_{k_1,1} & y_1 &= y_{0,1}a_{1,1}y_{1,1}\cdots a_{k_1,1}y_{k_1,1} \\ &\vdots & &\vdots \\ x_n &= x_{0,n}a_{1,n}x_{1,n}\cdots a_{k_n,n}x_{k_n,n} & y_n &= y_{0,n}a_{1,n}y_{1,n}\cdots a_{k_n,n}y_{k_n,n} \end{aligned} \tag{8}$$

we get two factorizations of $u$: $x_0b_1x_1\cdots b_nx_n$ and $y_0b_1y_1\cdots b_ny_n$. Since the product $L_0b_1L_1\cdots a_nL_n$ is unambiguous, we have $x_0 = y_0, \ldots, x_n = y_n$. Each of these words has now two factorizations given by (8) and since the products of (7) are unambiguous, these factorizations are equal. This proves the claim and the proposition. $\square$

We now consider the intersection of two unambiguous products.

**Theorem 5.3.** *If the products $K = K_0a_1K_1\cdots a_pK_p$ and $L = L_0b_1L_1\cdots b_qL_q$ are unambiguous, the products occurring in Formula* (4) *are all unambiguous.*

*Proof.* Let $(\alpha, \beta)$ be a pair of infiltration maps, and let $U_i = [K_{\bar\alpha(i)}]_{\uparrow z_i} \cap [L_{\bar\beta(i)}]_{\uparrow z_i}$, for $0 \leqslant i \leqslant r$. We claim that the product $U = U_0c_1U_1\cdots c_rU_r$ is unambiguous. Let

$$u = u_0c_1u_1\cdots c_ru_r = u'_0c_1u'_1\cdots c_ru'_r \tag{9}$$

be two factorizations of a word $u$ of $U$ such that, for $0 \leqslant i \leqslant r$, $u_i, u'_i \in U_i$. We prove by induction on $s$ that $u_s = u'_s$.

*Case $s = 0$.* By the properties of $\alpha$ and $\beta$, we may assume without loss of generality that $\alpha(1) = 1$, which implies that $c_1 = a_1$. It follows from ($C_3$) that $U_0 \subseteq K_0$. Now the product $K_0a_1(K_1a_2K_2\cdots a_pK_p)$ is unambiguous by Proposition 5.1, and by ($C_3$), $U_1c_2U_2\cdots c_rU_r$ is contained in $K_1a_1K_2\cdots a_pK_p$. Therefore, $u$ admits the two factorizations $u_0a_1(u_1c_2u_2\cdots c_ru_r)$ and $u'_0a_1(u'_1c_2u'_2\cdots c_ru'_r)$ in this product. Thus $u_0 = u'_0$.

*Induction step.* Let $s > 0$ and suppose by induction that $u_i = u'_i$ for $0 \leqslant i \leqslant s-1$. If $s = r$, then necessarily $u_s = u'_s$. If $s < r$, we may assume without loss of generality that $s$ is in the range of $\alpha$. Thus $\alpha(k) = s$ for some $k$ and $c_s = a_k$. We now consider two cases separately.

If $\alpha(k+1) = s+1$ (and $c_{s+1} = a_{k+1}$), it follows from ($C_3$) that $u$ has two factorizations

$$(u_0c_1u_1\cdots c_{s-1}u_{s-1})a_ku_sa_{k+1}(u_{s+1}c_{s+1}u_{s+2}\cdots c_ru_r) \text{ and}$$
$$(u_0c_1u_1\cdots c_{s-1}u_{s-1})a_ku'_sa_{k+1}(u'_{s+1}c_{s+1}u'_{s+2}\cdots c_ru'_r)$$

over the product $(K_0a_1K_1\cdots a_{s-1}K_{s-1})a_kK_sa_{k+1}(K_{s+1}a_{k+2}K_{s+2}\cdots a_pK_p)$. Since this product is unambiguous by Proposition 5.1, we get $u_s = u'_s$.

If $\alpha(k+1) \neq s+1$, then $s+1 = \beta(t+1)$ for some $t$ and $c_{s+1} = b_{t+1}$. Setting $i = \beta(t)$, we get $c_i = b_t$ and it follows from $(\mathrm{C}_4)$ that $u$ has two factorizations

$$(u_0c_1u_1\cdots c_{i-1}u_{i-1})b_t(u_ic_{i+1}u_{i+1}\cdots c_su_s)b_{t+1}(u_{s+1}c_{s+2}u_{s+2}\cdots c_ru_r) \text{ and}$$
$$(u_0c_1u_1\cdots c_{i-1}u_{i-1})b_t(u'_ic_{i+1}u'_{i+1}\cdots c_su'_s)b_{t+1}(u'_{s+1}c_{s+2}u'_{s+2}\cdots c_ru'_r)$$

over the product $(L_0b_1L_1\cdots b_{t-1}L_{t-1})b_tL_tb_{t+1}(L_{t+1}b_{t+1}L_{t+2}\cdots b_pL_p)$. This product is unambiguous by Proposition 5.1, and thus

$$u_ic_{i+1}u_{i+1}\cdots c_su_s = u'_ic_{i+1}u'_{i+1}\cdots c_su'_s$$

Now the induction hypothesis gives $u_i = u'_i, \ldots, u_{s-1} = u'_{s-1}$ and one finally gets $u_s = u'_s$. □

We state separately another interesting property.

**Theorem 5.4.** *Let $K = K_0a_1K_1\cdots a_pK_p$ and $L = L_0b_1L_1\cdots b_qL_q$ be two unambiguous products and let $(\alpha,\beta)$ and $(\alpha',\beta')$ be two pairs of infiltration maps of $I(p,q)$. If the sets $U(\alpha,\beta)$ and $U(\alpha',\beta')$ meet, then $\alpha = \alpha'$ and $\beta = \beta'$.*

*Proof.* Suppose that a word $u$ belongs to $U(\alpha,\beta)$ and to $U(\alpha',\beta')$. Then $u$ has two decompositions of the form

$$u = u_0c_1u_1\ \cdots\ c_ru_r = u'_0c'_1u'_1\ \cdots\ c'_{r'}u'_{r'}$$

Condition $(\mathrm{C}_1)$ $[(\mathrm{C}_2)]$ and the unambiguity of the product $K_0a_1K_1\cdots a_pK_p$ $[L_0b_1L_1\cdots b_qL_q]$ show that, for $0 \leqslant i \leqslant p$ and for $0 \leqslant j \leqslant q$,

$$\begin{aligned} &u_{\alpha(i)}c_{\alpha(i)+1}u_{\alpha(i)+1}\cdots c_{\alpha(i+1)-1}u_{\alpha(i+1)-1} = \\ &\qquad u'_{\alpha'(i)}c'_{\alpha'(i)+1}u'_{\alpha'(i)+1}\cdots c_{\alpha'(i+1)-1}u_{\alpha'(i+1)-1} \in K_i \end{aligned} \tag{10}$$

$$\begin{aligned} &u_{\beta(j)}c_{\beta(j)+1}u_{\beta(j)+1}\cdots c_{\beta(j+1)-1}u_{\beta(j+1)-1} = \\ &\qquad u'_{\beta'(j)}c'_{\beta'(j)+1}u'_{\beta'(j)+1}\cdots c_{\beta'(j+1)-1}u_{\beta'(j+1)-1} \in L_j \end{aligned} \tag{11}$$

We prove by induction on $s$ that, for $1 \leqslant s \leqslant \min(r,r')$, the following properties hold:

$\mathrm{E}_1(s)$ : $u_{s-1} = u'_{s-1}$ and $c_s = c'_s$,
$\mathrm{E}_2(s)$ : $\bar\alpha(s) = \bar\alpha'(s)$ and $\bar\beta(s) = \bar\beta'(s)$,
$\mathrm{E}_3(s)$ : for $i \leqslant \bar\alpha(s)$, $\alpha(i) = \alpha'(i)$ and for $j \leqslant \bar\beta(s)$, $\beta(j) = \beta'(j)$.

*Case* $s = 1$. We know that either $\alpha(1) = 1$ or $\beta(1) = 1$ and that either $\alpha'(1) = 1$ or $\beta'(1) = 1$. Suppose that $\alpha(1) = 1$. We claim that $\alpha'(1) = 1$. Otherwise, one has $\beta'(1) = 1$. Now, Formula (10) applied to $i = 0$ gives

$$u_0 = u'_0c'_1u'_1\cdots c_{\alpha'(1)-1}u'_{\alpha'(1)-1}$$

and Formula (11) applied to $j = 0$ gives

$$u_0 c_1 u_1 \cdots c_{\beta(1)-1} u_{\beta(1)-1} = u'_0.$$

Therefore $u_0 = u'_0$ and $\alpha'(1) = 1$, which proves the claim. It follows also that $a_1 = c_{\alpha(1)} = c_{\alpha'(1)}$ and thus $c_1 = c'_1$. We also have in this case $\bar{\alpha}(1) = \bar{\alpha}'(1) = 1$. A similar argument shows that if $\alpha'(1) = 1$, then $\alpha(1) = 1$. Therefore, the conditions $\alpha(1) = 1$ and $\alpha'(1) = 1$ are equivalent and it follows that $\bar{\alpha}(1) = \bar{\alpha}'(1)$. A dual argument would prove that the conditions $\beta(1) = 1$ and $\beta'(1) = 1$ are equivalent and that $\bar{\beta}(1) = \bar{\beta}'(1)$.

*Induction step.* Let $s$ be such that $1 \leqslant s+1 \leqslant \min(r, r')$ and suppose by induction that the properties $\mathrm{E}_1(i)$, $\mathrm{E}_2(i)$, $\mathrm{E}_3(i)$ hold for $1 \leqslant i \leqslant s$.

**Lemma 5.5.** *Suppose that $P_\alpha(s)$ holds and let $k = \bar{\alpha}(s)$. Then*

$$s \leqslant \alpha'(k+1) - 1 \tag{12}$$

*and*

$$u_s = u'_s c'_{s+1} u'_{s+1} \cdots c_{\alpha'(k+1)-1} u'_{\alpha'(k+1)-1} \tag{13}$$

*Proof.* Applying (10) with $i = k$, we get

$$u_{\alpha(k)} c_{\alpha(k)+1} u_{\alpha(k)+1} \cdots c_s u_s = u'_{\alpha'(k)} c'_{\alpha'(k)+1} u'_{\alpha'(k)+1} \cdots c_{\alpha'(k+1)-1} u'_{\alpha'(k+1)-1} \tag{14}$$

Since $\bar{\alpha}(s) = \bar{\alpha}'(s)$ by $\mathrm{E}_2(s)$, one has $\bar{\alpha}'(s) = k$ and $\alpha'(k+1) \geqslant s+1$ by Lemma 3.1, which gives (12). Further, since $k = \bar{\alpha}(s)$, it follows from $\mathrm{E}_3(s)$ that $\alpha(k) = \alpha'(k)$. Now, for $i \leqslant s$, $\mathrm{E}_1(i)$ implies that $u_{i-1} = u'_{i-1}$ and $c_i = c'_i$. It follows that the word $u_{\alpha(k)} c_{\alpha(k)+1} u_{\alpha(k)+1} \cdots c_s$ is a prefix of both sides of (14). Therefore, this prefix can be deleted from both sides of (14), which gives (13). □

We now establish $\mathrm{E}_1(s+1)$.

**Lemma 5.6.** *One has $u_s = u'_s$ and $c_{s+1} = c'_{s+1}$. Further, $P_\alpha(s)$ and $P_{\alpha'}(s)$ are equivalent and $P_\beta(s)$ and $P_{\beta'}(s)$ are equivalent.*

*Proof.* Let us prove that $u'_s$ is a prefix of $u_s$. By Lemma 3.3, either $P_\alpha(s)$ or $P_\beta(s)$ holds. Suppose that $P_\alpha(s)$ holds. Then by Lemma 5.5, $u'_s$ is a prefix of $u_s$. If $P_\beta(s)$ holds, we arrive to the same conclusion by using (11) in place of (10) in the proof of Lemma 5.5.

Now, a symmetrical argument using the pair $(\bar{\alpha}', \bar{\beta}')$ would show that $u_s$ is a prefix of $u'_s$. Therefore, $u_s = u'_s$. Coming back to (13), we obtain $\alpha'(k+1) = s+1$ and since by $\mathrm{E}_2(s)$, $k = \bar{\alpha}(s) = \bar{\alpha}'(s)$, one gets $\alpha'(\bar{\alpha}'(s)+1) = s+1$, which, by Lemma 3.2, is equivalent to $P_{\alpha'}(s)$. Thus $P_\alpha(s)$ implies $P_{\alpha'}(s)$ and a dual argument would prove the opposite implication.

We also have $c_{s+1} = c_{\alpha(k+1)} = a_{k+1} = c'_{\alpha'(k+1)} = c'_{s+1}$ and thus $c_{s+1} = c'_{s+1}$. Finally, a similar argument works for $\beta$. □

We now come to the proof of $\mathrm{E}_2(s+1)$ and $\mathrm{E}_3(s+1)$. Since $P_\alpha(s)$ and $P_{\alpha'}(s)$ are equivalent, the next two lemma cover all cases.

**Lemma 5.7.** *If neither $P_\alpha(s)$ nor $P_{\alpha'}(s)$ hold, then $\bar\alpha(s+1) = \bar\alpha'(s+1)$ and for $i \leqslant \bar\alpha(s+1)$, $\alpha(i) = \alpha'(i)$. Similarly, if neither $P_\beta(s)$ nor $P_{\beta'}(s)$ hold, then $\bar\beta(s+1) = \bar\beta'(s+1)$ and for $i \leqslant \bar\beta(s+1)$, $\beta(i) = \beta'(i)$.*

*Proof.* We just prove the "$\alpha$ part" of the lemma. If neither $P_\alpha(s)$ nor $P_{\alpha'}(s)$ hold, then $\bar\alpha(s+1) = \bar\alpha(s)$ and $\bar\alpha'(s+1) = \bar\alpha'(s)$. Since $\bar\alpha(s) = \bar\alpha'(s)$ by $\mathrm{E}_2(s)$, one gets $\bar\alpha(s+1) = \bar\alpha'(s+1)$. The second property is an immediate consequence of $\mathrm{E}_3(s)$. $\square$

**Lemma 5.8.** *If both $P_\alpha(s)$ and $P_{\alpha'}(s)$ hold, then $\bar\alpha(s+1) = \bar\alpha'(s+1)$ and for $i \leqslant \bar\alpha(s+1)$, $\alpha(i) = \alpha'(i)$. Similarly, if both $P_\beta(s)$ and $P_{\beta'}(s)$ hold, then $\bar\beta(s+1) = \bar\beta'(s+1)$ and for $i \leqslant \bar\beta(s+1)$, $\beta(i) = \beta'(i)$.*

*Proof.* Again, we just prove the "$\alpha$ part" of the lemma. If both $P_\alpha(s)$ and $P_{\alpha'}(s)$ hold, then $\bar\alpha(s+1) = \bar\alpha(s)+1$ and $\bar\alpha'(s+1) = \bar\alpha'(s)+1$. Since $\bar\alpha(s) = \bar\alpha'(s)$ by $\mathrm{E}_2(s)$, one gets $\bar\alpha(s+1) = \bar\alpha'(s+1)$. Property $\mathrm{E}_3(s)$ shows that for $i \leqslant \bar\alpha(s)$, $\alpha(i) = \alpha'(i)$. Since $\bar\alpha(s+1) = \bar\alpha(s)+1$, it just remains to prove that

$$\alpha(\bar\alpha(s+1)) = \alpha'(\bar\alpha(s+1)) \tag{15}$$

But Lemma 3.2 shows that $\alpha(\bar\alpha(s+1)) = s+1$ and $\alpha'(\bar\alpha'(s+1)) = s+1$, which proves (15) since $\bar\alpha(s+1) = \bar\alpha'(s+1)$. $\square$

This concludes the induction step and the proof of Theorem 5.4. $\square$

**Corollary 5.9.** *Let $\mathcal{L}$ be a lattice of regular languages closed under quotient. Then its unambiguous polynomial closure is also a lattice closed under quotient.*

If $\mathcal{L}$ is a Boolean algebra, then one can be more precise.

**Corollary 5.10.** *Let $\mathcal{L}$ be a Boolean algebra of regular languages closed under quotient. Then its unambiguous polynomial closure is also a Boolean algebra closed under quotient.*

Let us conclude with an example which shows that, under the assumptions of Theorem 5.4, the sets $U(\alpha,\beta)$ cannot be further decomposed as a disjoint union of unambiguous products.
Let $K = K_0aK_1$ and $L = L_0aL_1$ with $K_0 = L_1 = 1+b+c+c^2$ and $L_0 = K_1 = a+ab+ba+ac+ca+ac^2+bab+cac+cac^2$. Then

$$\begin{aligned}K \cap L = aa + aab + aba + aac + aca + aac^2 + abab + acac + acac^2 +\\ baa + baab + baba + baac + baac^2 + babab + caa +\\ caab + caac + caca + caac^2 + cacac + cacac^2\end{aligned}$$

One can write for instance $K \cap L$ as $(1+b+c)aa(1+b+c+c^2) + (1+b)a(1+b)a(1+b) + (1+c)a(1+c)a(1+c+c^2)$ but the three components of this language are not disjoint, since they all contain $aa$. Note that the words $acab$, $abac$, $baca$ and $caba$ are not in $K \cap L$.

The syntactic ordered monoid of $K_0$ and $L_1$ has 4 elements $\{1, a, b, c\}$ and is presented by the relations $a = ba = b^2 = bc = ca = cb = 0$ and $c^2 = b$. Its syntactic order is defined by $a < b < c < 1$.

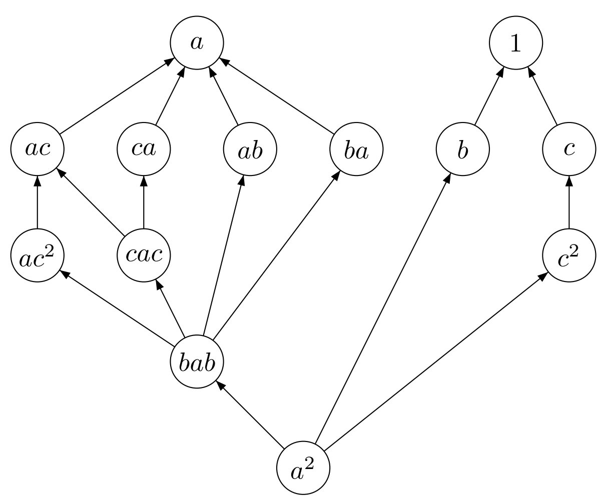
The syntactic ordered monoid of $L_0$ and $K_1$ has 13 elements:

$$\{1, a, b, c, a^2, ab, ac, ba, ca, c^2, ac^2, bab, cac\}$$

and is defined by the relations $cac^2 = bab$ and

$$b^2 = bc = cb = a^2 = aba = aca = bac = cab = c^2a = c^3 = 0.$$

The syntactic order is:




There is only one pair of infiltration maps $(\alpha, \beta)$ of $I(1, 1)$ that defines a nonempty set $U(\alpha, \beta)$. This pair is defined as follows: $\alpha(1) = 1$ and $\beta(1) = 2$. The triples $(z_0, z_1, z_2)$ of $C(\alpha, \beta)$ are exactly the triples of words such that $z_0 a z_1 a z_2 \in K \cap L$. In particular, $z_0 \in \{1, b, c\}$, $z_1 \in \{1, b, c\}$ and $z_2 \in \{1, b, c, c^2\}$. Now, one has

$$\begin{array}{llll}
[K_0]_{\uparrow 1} = 1 & [K_0]_{\uparrow b} = 1 + b + c + c^2 & [K_0]_{\uparrow c} = 1 + c & \\
[K_1]_{\uparrow 1} = 1 & [K_1]_{\uparrow b} = 1 + b & [K_1]_{\uparrow c} = 1 + c & [K_1]_{\uparrow c^2} = 1 + c + c^2 \\
[L_0]_{\uparrow 1} = 1 & [L_0]_{\uparrow b} = 1 + b & [L_0]_{\uparrow c} = 1 + c & \\
[L_1]_{\uparrow 1} = 1 & [L_1]_{\uparrow b} = 1 + b + c + c^2 & [L_1]_{\uparrow c} = 1 + c & [L_1]_{\uparrow c^2} = 1 + b + c + c^2
\end{array}$$

which gives the following possibilities for the triples $(U_0, U_1, U_2)$, for the following triples $z = (z_0, z_1, z_2)$:

$$
\begin{array}{llll}
z = (1,1,1) & U_0 = 1 & U_1 = 1 & U_2 = 1 \\
z = (b,b,b) & U_0 = 1+b & U_1 = 1+b & U_2 = 1+b \\
z = (c,c,c) & U_0 = 1+c & U_1 = 1+c & U_2 = 1+c \\
z = (b,c,c^2) & U_0 = 1+b & U_1 = 1+c & U_2 = 1+c+c^2 \\
z = (c,c,c^2) & U_0 = 1+c & U_1 = 1+c & U_2 = 1+c+c^2
\end{array}
$$

### 5.2 Deterministic product

The marked product $L = L_0a_1L_1 \cdots a_nL_n$ of $n$ nonempty languages $L_0$, $L_1$, …, $L_n$ of $A^*$ is *left deterministic* [*right deterministic*] if, for $1 \leqslant i \leqslant n$, the set $L_0a_1L_1 \cdots L_{i-1}a_i$ $[a_iL_i \cdots a_nL_n]$ is a prefix [suffix] code. This means that every word of $L$ has a unique prefix [suffix] in $L_0a_1L_1 \cdots L_{i-1}a_i$ $[a_iL_i \cdots a_nL_n]$. It is observed in [3, p. 495] that the marked product $L_0a_1L_1 \cdots a_nL_n$ is deterministic if and only if, for $1 \leqslant i \leqslant n$, the language $L_{i-1}a_i$ is a prefix code. Since the product of two prefix codes is a prefix code, we get the following proposition.

**Proposition 5.11.** *Any left [right] deterministic product of left [right] deterministic products is left [right] deterministic.*

*Proof.* This follows immediately from the fact that the product of two prefix codes is a prefix code. □

Factorizing a deterministic product also gives a deterministic product. More precisely, one has the following result.

**Proposition 5.12.** *Let $L_0a_1L_1 \cdots a_nL_n$ be a left [right] deterministic product and let $i_1$, …, $i_k$ be a sequence of integers satisfying $0 < i_1 < \ldots < i_k < n$. Finally, let $R_0 = L_0a_1L_1 \cdots a_{i_1-1}L_{i_1-1}$, …, $R_k = L_{i_k}a_{i_k+1}L_{i_k+1} \cdots L_{n-1}a_nL_n$. Then the product $R_0a_{i_1}R_1 \cdots a_{i_k}R_k$ is left [right] deterministic.*

*Proof.* Trivial. □

The *left [right] deterministic polynomial closure* of a class of languages $\mathcal{L}$ of $A^*$ is the set of languages that are finite unions of left [right] deterministic products of the form $L_0a_1L_1 \cdots a_nL_n$, where the $a_i$'s are letters and the $L_i$'s are elements of $\mathcal{L}$.

We can now state the counterpart of Theorem 5.3 for deterministic products.

**Theorem 5.13.** *If the products $K = K_0a_1K_1 \cdots a_pK_p$ and $L = L_0b_1L_1 \cdots b_qL_q$ are deterministic, the products occurring in Formula (4) are all deterministic.*

*Proof.* Let $i \in \{0, \ldots, r\}$. By construction, there exists $k \geqslant 0$ such that $i+1 = \alpha(k+1)$ or $i+1 = \beta(k+1)$. By Lemma 4.2, there exists $j \leqslant i$ such that either $U_jc_{j+1}U_{j+1} \cdots U_i \subseteq K_k$ and $c_{\alpha(k+1)} = a_{k+1}$ or $U_jc_{j+1}U_{j+1} \cdots U_i \subseteq L_k$ and

$c_{\alpha(k+1)} = b_{k+1}$. Suppose we are in the first case and that $U_i c_{i+1}$ is not a prefix code. Then $U_j c_{j+1} U_{j+1} \cdots U_i c_{i+1}$ is not a prefix code and thus $K_k a_{k+1}$ is not a prefix code. This yields a contradiction since the product $K_0 a_1 K_1 \cdots a_p K_p$ is deterministic. $\square$

**Corollary 5.14.** *Let $\mathcal{L}$ be a lattice of regular languages closed under quotient. Then its deterministic polynomial closure is also closed under quotient.*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## References


1. M. Arfi, Opérations polynomiales et hiérarchies de concaténation, *Theoret. Comput. Sci.* **91** (1991), 71–84.
2. M. J. J. Branco, On the Pin-Thérien expansion of idempotent monoids, *Semigroup Forum* **49**,3 (1994), 329–334.
3. M. J. J. Branco, The kernel category and variants of the concatenation product, *Internat. J. Algebra Comput.* **7**,4 (1997), 487–509.
4. M. J. J. Branco, Two algebraic approaches to variants of the concatenation product, *Theoret. Comput. Sci.* **369**,1-3 (2006), 406–426.
5. M. J. J. Branco, Deterministic concatenation product of languages recognized by finite idempotent monoids, *Semigroup Forum* **74**,3 (2007), 379–409.
6. M. J. J. Branco and J.-E. Pin, Equations for the polynomial closure, in *ICALP 2009, Part II*, S. Albers and W. Thomas (eds.), pp. 115–126, *Lect. Notes Comp. Sci.* vol. 5556, Springer, Berlin, 2009.
7. M. Lothaire, *Combinatorics on words*, *Cambridge Mathematical Library*, Cambridge University Press, Cambridge, 1997.
8. D. Perrin and J.-E. Pin, *Infinite Words*, *Pure and Applied Mathematics* vol. 141, Elsevier, 2004. ISBN 0-12-532111-2.
9. J.-E. Pin, Propriétés syntactiques du produit non ambigu, in *7th ICALP*, pp. 483–499, *Lect. Notes Comp. Sci.* n° 85, Springer, Berlin, 1980.
10. J.-E. Pin, A variety theorem without complementation, *Russian Mathematics (Iz. VUZ)* **39** (1995), 80–90.
11. J.-E. Pin, H. Straubing and D. Thérien, Locally trivial categories and unambiguous concatenation, *J. of Pure and Applied Algebra* **52** (1988), 297–311.
12. J.-E. Pin and D. Thérien, The bideterministic concatenation product, *Internat. J. Algebra Comput.* **3** (1993), 535–555.
13. J.-E. Pin and P. Weil, Polynomial closure and unambiguous product, *Theory Comput. Systems* **30** (1997), 1–39.
14. M.-P. Schützenberger, Une théorie algébrique du codage, in *Séminaire Dubreil-Pisot, année 1955-56, Exposé No. 15, 27 février 1956, 24 pages*, Inst. H. Poincaré, Paris, 1956.
15. M.-P. Schützenberger, Sur le produit de concaténation non ambigu, *Semigroup Forum* **18** (1976), 331–340.